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Shells, thin cylindrical

Applied mechanics

A METHOD OF SUCCESSIVE APPROXIMATION APPLIED TO A CLASS OF SHELLS OF REVOLUTION WITH REGIONS OF RAPIDLY VARYING THICKNESS

Technical Report WAL TR 893.3/4

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Date of Issue - March 1963

AMS Code 5011.11.838
Basic Research in Physical Sciences

D/A Project 1-A-0-10501-B-010

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TITLE

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ABSTRACT

A plan of successive approximations is outlined for handling the equations of the three-dimensional problem in elasticity for shells of revolution with regions of rapidly varying thickness along the conceptual lines of a technique proposed by O. Göhner. Attention is confined to essentially cylindrical shells with regions of rapidly varying thickness, e.g., circumferential notches or grooves. For a restricted but useful class of loadings, plane biharmonic stress functions can be utilized. The first two orders of theory are explicitly formulated in the framework of analytic functions of a complex variable.

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I. INTRODUCTION

Occasionally, the need arises for the analysis of moderately thick shells with rapidly varying wall thickness. Local stress concentration becomes the primary concern in certain cases rather than the averaged behavior approximated by conventional thin shell analysis. Clearly the determination of such information requires a careful consideration of the equations of three-dimensional elasticity theory.

In this report, we initially consider the class of moderately thick shells of revolution under axially symmetric loadings, i.e., the deformation will be axially symmetric. The local effects of geometrical irregularities in the wall thickness (e.g., circumferential grooves or nodes) will be of principal concern. Although the corresponding problem class in three-dimensional elasticity is well-known, for example Reference 1, exact solutions in general are extremely difficult to obtain. The plan of analysis therefore consists in utilizing the relative thinness of the shell to systematically approximate the three-dimensional equations.

A plan of successive approximations, conceptually due to O. Göhner,[2] is adopted. The choice of the order of approximation is obviously sensitive to the loadings considered. A particular choice is considered in this report leading to a structure permitting the use of plane biharmonic functions.

It is shown that this plan is suitable for handling a restricted but useful class of loadings of cylindrical shells with rapidly varying thickness. In particular, the zero'th order solution supports Neuber's assertion[3] that when the shell is very thin for a class of applied loadings the problem can be considered as one of plane strain. The first two orders of solution can be explicitly formulated in the framework of analytic functions of a complex variable.

II. FORMULATION IN TERMS OF SUCCESSIVE APPROXIMATIONS

For the problem class of axially symmetric deformation, it will be convenient initially to review the formulation in terms of cylindrical coordinates (r, θ, z) where z is the axis of revolution. Assuming no body forces, it is well-known, e.g., Reference 1, that the equations of equilibrium are

$$\begin{aligned} \sigma_{r,r} + \tau_{rz,z} + (\sigma_r - \sigma_\theta)/r &= 0 \\ \tau_{rz,r} + \sigma_{z,z} + \tau_{rz}/r &= 0 \end{aligned} \tag{1}$$

where an independent variable appearing as a subscript and following a comma denotes partial differentiation with respect to that variable. The compatibility equations for this case are

$$\nabla^2\sigma_r - 2(\sigma_r - \sigma_\theta)/r^2 + \Omega,_{rr}/(1+\nu) = 0,$$
$$\nabla^2\sigma_\theta + 2(\sigma_r - \sigma_\theta)/r^2 + \Omega,_r/r(1+\nu) = 0, \tag{2}$$
$$\nabla^2\sigma_z + \Omega,_{zz}/(1+\nu) = 0,$$
$$\nabla^2\tau_{rz} - \tau_{rz}/r^2 + \Omega,_{rz}/(1+\nu) = 0;$$

where

$$\nabla^2 = \partial^2/\partial r^2 + (1/r)\partial/\partial r + \partial^2/\partial z^2,$$

$$\Omega = (\sigma_r + \sigma_\theta + \sigma_z)/(1+\nu),$$

and ν denotes Poisson's ratio.

The systems (1) and (2) can be replaced by a formulation in terms of a single stress function, ϕ, as proposed by Love,[4] by defining

$$\sigma_r = [\nu\nabla^2\phi - \phi_{rr}]_z,$$
$$\sigma_\theta = [\nu\nabla^2\phi - \phi_r/r]_z, \tag{3}$$
$$\sigma_z = [(2-\nu)\nabla^2\phi - \phi_{zz}]_z,$$
$$\tau_{rz} = [(1-\nu)\nabla^2\phi - \phi_{zz}]_r,$$

where

$$\nabla^2\nabla^2\phi = 0.$$

(In (3), subscript notation denotes partial differentation of the stress function $\phi(r, z)$.)

Let us now introduce the new coordinates ξ, η defined by

$$\xi = r - R, \quad \eta = z, \tag{4}$$

where R is an appropriately chosen fixed radius of the shell. For the present problem class, i.e., cylinders of variable thickness, we consider R as independent of r and z. The previous systems can now be considered in terms of the ξ, η coordinates. For example, the equations of equilibrium and compatibility have the forms

$$\sigma_{\xi,\xi} + \tau_{\xi\eta,\eta} + (\sigma_\xi - \sigma_\theta)/(\xi + R) = 0, \tag{1'}$$

$$\Delta\sigma_\xi + \sigma_{\xi,\xi}/(\xi + R) - 2(\sigma_\xi - \sigma_\theta)/(\xi + R)^2 + \Omega,_{\xi\xi}/(1 + \nu) = 0, \qquad (2')$$

where

$$\Delta = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2.$$

Göhner's plan of successive approximations depends on the validity of the expressions

$$1/(\xi + R) = (1/R) \sum_{n=0}^{\infty} (- \xi/R)^n,$$

$$1/(\xi + R)^2 = (1/R)^2 \sum_{n=0}^{\infty} (n+1)(- \xi/R)^n. \qquad (5)$$

We shall henceforth confine our attention to regions where the expansions (5) can be considered valid. Thus, the expansions (5) will be considered in (1') and (2') and a plan of successive approximations to the system will be defined. In particular, if σ is the generic symbol for the stresses, it is assumed that

$$\sigma = \sum_{n=0}^{\infty} \sigma^{(n)}, \qquad (6)$$

where the superscript denotes the order of the approximation.

In the method of successive approximations, the plan of approximation is clearly not unique. Furthermore, a rigorous assessment of the validity of any given plan is generally unfeasible. On the other hand, several minimal requirements are obvious. The plan should lead to a systematic satisfaction of equilibrium and compatibility at least in an asymptotic sense. Furthermore, each order of solution should correspond to a well-defined mathematical system of equations. Finally, a reasonably accurate approximation of the prescribed load conditions should be possible.

We propose the following particular plan of successive approximation: For the n'th order approximation with leading components $\sigma^{(n)}$, terms containing ξ^n/R^{n+1} explicitly in (1') and (2') will be neglected. Thus, the zero'th order approximation corresponds to the system

$$\sigma^{(0)}_{\xi,\xi} + \tau^{(0)}_{\xi\eta,\eta} = 0, \quad \tau^{(0)}_{\xi\eta,\xi} + \sigma^{(0)}_{\eta,\eta} = 0,$$

$$\Delta\sigma^{(0)}_\xi + \Omega^{(0)},_{\xi\xi}/(1 + \nu) = 0, \quad \Delta\sigma^{(0)}_\theta = 0, \qquad (7)$$

$$\Delta\sigma^{(0)}_\eta + \Omega^{(0)},_{\eta\eta}/(1 + \nu) = 0, \quad \Delta\tau^{(0)}_{\xi\eta} + \Omega^{(0)},_{\xi\eta}/(1 + \nu) = 0.$$

For n = 1, the corresponding system is

$$
\begin{aligned}
\sigma^{(1)}_{\xi,\xi} + \tau^{(1)}_{\xi\eta,\eta} + (\sigma^{(0)}_{\xi} - \sigma^{(0)}_{\theta})/R &= 0, \\
\tau^{(1)}_{\xi\eta,\xi} + \sigma^{(1)}_{\eta,\eta} + \tau^{(0)}_{\xi\eta}/R &= 0,
\end{aligned}
\tag{8}
$$

etc.

The plan can be conveniently summarized in terms of stress functions. Guided by the structure of the stress function (3), it is a straightforward matter to determine a system of stress functions compatible with the preceding plan of successive approximation. In fact, the n'th order approximation can be replaced by

$$
\begin{aligned}
\sigma^{(n)}_{\xi} &= \left[\nu\Delta F^{(n)} - F^{(n)}_{\xi\xi} + (\nu/R)\sum_{j=1}^{n} F^{(j-1)}_{\xi}(-\xi/R)^{n-j}\right]_{\eta}, \\
\sigma^{(n)}_{\theta} &= \left[\nu\Delta F^{(n)} - (1-\nu)\sum_{j=1}^{n} F^{(j-1)}_{\xi}(-\xi/R)^{n-j}/R\right]_{\eta}, \\
\sigma^{(n)}_{\eta} &= \left[(1-\nu)\Delta F^{(n)} + F^{(n)}_{\xi\xi} + (2-\nu)\sum_{j=1}^{n} F^{(j-1)}_{\xi}(-\xi/R)^{n-j}/R\right]_{\eta}, \\
\tau^{(n)}_{\xi\eta} &= \left[F^{(n)}_{\xi\xi} - \nu\Delta F^{(n)} + (1-\nu)\sum_{j=1}^{n} F^{(j-1)}_{\xi}(-\xi/R)^{n-j}/R\right]_{\xi},
\end{aligned}
\tag{9}
$$

$$n = 0,1,2,\ldots.$$

where $F^{(i)}(\xi,\eta) = 0$ if $i<0$ and

$$
\begin{aligned}
&\Delta\Delta F^{(n)} + (2/R)\sum_{i=0}^{n-1}\Delta F^{(n-i-1)}_{\xi}(-\xi/R)^{i} \\
&-(1/R^2)\sum_{i=0}^{n-2}(i+1)F^{(n-i-2)}_{\xi\xi}(-\xi/R)^{i} + (1/2R^2)\sum_{i=0}^{n-2}(i+2)(i+1)F^{(n-i-2)}_{\xi}(-\xi/R)^{i} = 0,
\end{aligned}
\tag{10}
$$

$$n = 0,1,2,\ldots.$$

The plan of approximation is not yet complete. Although a procedure has been invented for the systematic consideration of equilibrium and compatibility, there is no assurance that each system is well-defined and that reasonable approximation of the applied loads can be realised. In order to study the complete structure of each system, it is necessary to consider the matter of boundary conditions to be imposed on the individual systems. Therefore the applied load or possible systematic approximations of the applied load must be investigated.

III. THE ZERO'TH ORDER THEORY

For $n = 0$, (9) and (10) yield

$$\begin{aligned}
\sigma_\xi^{(0)} &= \left[\nu\Delta F^{(0)} - F_{\xi\xi}^{(0)}\right]_\eta, \\
\sigma_\theta^{(0)} &= \nu\left[\sigma_\xi^{(0)} + \sigma_\eta^{(0)}\right], \\
\sigma_\eta^{(0)} &= \left[(1-\nu)\Delta F^{(0)} + F_{\xi\xi}^{(0)}\right]_\eta, \\
\tau_{\xi\eta}^{(0)} &= \left[F_{\xi\xi}^{(0)} - \nu\Delta F^{(0)}\right]_\xi,
\end{aligned} \tag{11}$$

where

$$\Delta\Delta F^{(0)} = 0. \tag{12}$$

Although this system is apparently consistent with the well-known problem of plane strain, such a conclusion cannot be immediately arrived at without further consideration of the applied load conditions.

In Figure 1, the force resultants T_η ds and T_ξ ds per unit of circumference acting on the arc ds is shown where it is assumed that the force is exerted by the material to the left on the material to the right proceeding from A′ to B′.

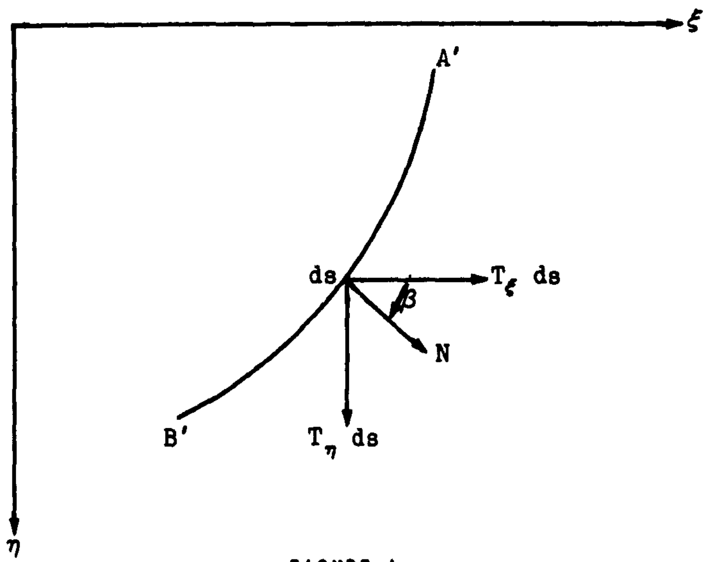

FIGURE 1.

From equilibrium,

$$T_\eta = \sigma_\eta \sin\beta + \tau_{\xi\eta} \cos\beta = -\sigma_\eta \, d\xi/ds + \tau_{\xi\eta} \, d\eta/ds, \tag{13}$$

$$T_\xi = \sigma_\xi \cos\beta + \tau_{\xi\eta} \sin\beta = \sigma_\xi \, d\eta/ds - \tau_{\xi\eta} \, d\xi/ds. \tag{14}$$

We now define as an additional part of our plan of successive approximations,

$$T_\eta = \sum_{j=0}^{\infty} T_\eta^{(j)}, \quad T_\xi = \sum_{j=0}^{\infty} T_\xi^{(j)}, \tag{15}$$

where

$$\begin{aligned} T_\eta^{(n)} &= -\sigma_\eta^{(n)} \, d\xi/ds + \tau_{\xi\eta}^{(n)} \, d\eta/ds, \quad n = 0,1,2,\ldots , \\ T_\xi^{(n)} &= \sigma_\xi^{(n)} \, d\eta/ds - \tau_{\xi\eta}^{(n)} \, d\xi/ds, \quad n = 0,1,2, \ldots . \end{aligned} \tag{16}$$

Thus,

$$T_\eta^{(0)} = - d\, F_{\xi\eta}^{(0)}/ds - (1 - \nu)\left[\Delta F_\eta^{(0)} \, d\xi/ds - \Delta F_\xi^{(0)} \, d\eta/ds\right], \tag{17}$$

$$T_\xi^{(0)} = - d\left[F_{\xi\xi}^{(0)} - \nu\Delta F^{(0)}\right]/ds. \tag{18}$$

Consider now the resultant forces, R_η and R_ξ acting on the surface generated by the arc A'B'. R_ξ vanishes by symmetry, whereas

$$R_\eta = 2\pi \int_{A'}^{B'} (R + \xi) \, T_\eta \, ds. \tag{19}$$

If the arc A'B' is a closed contour with respect to the $\xi\eta$-plane, then R_η should vanish as B' coincides with A' from a consideration of equilibrium. If T_η is approximated by $T_\eta^{(0)}$ and substituted in (19), it is easily shown that R_η does not vanish in general for a closed circuit. It is therefore clear that a meaningful definition of R_η must be given to each order of theory allowing for the order of approximation to the equilibrium state. A similar argument can be made for the displacements and the resultant moments.

A well-defined zero'th order theory will now be considered. We assume the stresses of the zero'th order theory are defined by (11) and (12). The function $F^{(0)}(\xi,\eta)$ and its fourth order partial derivatives are therefore assumed to exist at all interior points of the region described in the $\xi\eta$-plane. Consider

$$P_\eta^{(0)} = \int_{A'}^{B'} T_\eta^{(0)} \, ds, \tag{20}$$

$$P_\xi^{(0)} = \int_{A'}^{B'} T_\xi^{(0)} \, ds, \tag{21}$$

where $T_\eta^{(0)}$ and $T_\xi^{(0)}$ are defined by (17) and (18). For any closed contour A'B' (bounding a simply connected region of the material in the $\xi\eta$-plane), $P_\eta^{(0)}$ and $P_\eta^{(0)}$ vanish due to the continuity of $F^{(0)}$ and its third order partial derivatives. Thus, (11) and (12) coupled with (17) and (18) imply a system of applied loads whose resultant force vanishes from a plane point of view. A similar result follows when the resultant moment of the applied load is considered.

It has thus been shown that for the plan of successive approximation outlined above, the zero'th order approximation implies a system of applied loads which is *self-equilibrating in the plane sense*. It would be inconsistent therefore to attempt to use this particular plan of approximation for loading systems which violate this condition to the first order, e.g., internal pressure acting on a cylindrical shell. Therefore, we henceforth restrict the class of applied loads to those which are self-equilibrating in the plane sense to the first order.

Several details are still necessary to correlate the zero'th order theory with the original three-dimensional problem. The force resultant R_η defined by (19) will now be considered as

$$R_\eta = \sum_{j=0}^{\infty} R_\eta^{(j)}. \tag{22}$$

Therefore, a compatible definition of $R_\eta^{(0)}$ with the zero'th order theory is

$$R_\eta^{(0)} = 2\pi R \int_{A'}^{B'} T_\eta^{(0)} \, ds. \tag{23}$$

Finally, we assume the radial and axial displacement components u and w can be expressed as

$$u = \sum_{j=0}^{\infty} u^{(j)}, \quad w = \sum_{j=0}^{\infty} w^{(j)}. \tag{24}$$

It will be recalled that for the zero'th order theory,

$$\sigma_\theta^{(0)} = \nu(\sigma_\xi^{(0)} + \sigma_\eta^{(0)}). \tag{25}$$

Therefore, from Hooke's law and the well-known strain-displacement relations in cylindrical coordinates,

$$E\ \partial u^{(0)}/\partial\xi = (1-\nu^2)\sigma_\xi^{(0)} - \nu(1+\nu)\sigma_\eta^{(0)},$$

$$E\ \partial w^{(0)}/\partial\eta = (1-\nu^2)\sigma_\eta^{(0)} - \nu(1+\nu)\sigma_\xi^{(0)}, \tag{26}$$

$$G(\partial u^{(0)}/\partial\eta + \partial w^{(0)}/\partial\xi) = \tau_{\xi\eta}^{(0)},$$

where E is Young's modulus and $G = E/2(1+\nu)$. Thus,

$$2G\ \partial u^{(0)}/\partial\xi = -F_{\xi\xi\eta}^{(0)}, \tag{27}$$

$$2G\ \partial w^{(0)}/\partial\eta = (1-2\nu)\Delta F_\eta^{(0)} + F_{\xi\xi\eta}^{(0)}.$$

Apart from rigid body motions,

$$2G\ u^{(0)} = -F_{\xi\eta}^{(0)},$$

$$2G\ w^{(0)} = (1-2\nu)\Delta F^{(0)} + F_{\xi\xi}^{(0)}. \tag{28}$$

IV. THE ORDER OF THEORY, n = 1

For orders of theory $n \geq 1$, the stress functions $F^{(n)}(\xi,\eta)$ are no longer plane biharmonic in general. On the other hand, the application of plane biharmonic functions can be preserved provided a particular integral of (10) is found for each order of theory. In general, the determination of the particular integrals requires a knowledge of the preceding orders of solution.

Fortunately, the particular integral for the order $n = 1$ can be found directly. In fact, for $n = 1$, $(-\xi/R)F^{(0)}/2$ is an integral of (10). Therefore, the function $G^{(1)}(\xi,\eta)$ is defined as

$$F^{(1)} = G^{(1)} - \xi F^{(0)}/2R. \tag{29}$$

Then,

$$\sigma_\xi^{(1)} = \left[\nu\Delta G^{(1)} - G_{\xi\xi}^{(1)} - \nu\xi\Delta F^{(0)}/2R + \xi F_{\xi\xi}^{(0)}/2R + F_\xi^{(0)}/R\right]_{\eta},$$

$$\sigma_\theta^{(1)} = \left[\nu\Delta G^{(1)} - \nu\xi\Delta F^{(0)}/2R - F_\xi^{(0)}/R\right]_{\eta}, \tag{30}$$

$$\sigma_\eta^{(1)} = \left[(1-\nu)\Delta G^{(1)} + G_{\xi\xi}^{(1)} - (1-\nu)\xi\Delta F^{(0)}/2R - \xi F_{\xi\xi}^{(0)}/2R\right]_{\eta},$$

$$\tau_{\xi\eta}^{(1)} = \left[G_{\xi\xi}^{(1)} - \nu\Delta G^{(1)} - \xi F_{\xi\xi}^{(0)}/2R + \nu\xi\Delta F^{(0)}/2R\right]_{\xi},$$

and

$$\Delta\Delta\, G^{(1)} = 0. \tag{31}$$

The corresponding stress resultants are

$$T_\eta^{(1)} = -\sigma_\eta^{(1)} d\xi/ds + \tau_{\xi\eta}^{(1)} d\eta/ds$$

$$= -dG_{\xi\eta}^{(1)}/ds - (1-\nu)\left[\Delta G_\eta^{(1)} d\xi/ds - \Delta G_\xi^{(1)} d\eta/ds\right] \tag{32}$$

$$- \xi T_\eta^{(0)}/2R - \left[F_{\xi\xi}^{(0)} - \nu\Delta F^{(0)}\right] d\eta/2Rds,$$

$$T_\xi^{(1)} = \sigma_\xi^{(1)} d\eta/ds - \tau_{\xi\eta}^{(1)} d\xi/ds$$

$$= -d\left[G_{\xi\xi}^{(1)} - \nu\Delta G^{(1)}\right]/ds - \xi T_\xi^{(0)}/2R \tag{33}$$

$$+ F_{\xi\eta}^{(0)} d\eta/Rds + \left[F_{\xi\xi}^{(0)} - \nu\Delta F^{(0)}\right] d\xi/2Rds.$$

An interesting mathematical dilemma occurs from a consideration of the determinancy of the solution. In general, a logical definition of $T_\eta^{(1)}$ and $T_\xi^{(1)}$ would lead to a plane self-equilibrating set of prescribed loads. For example, if $T_\xi^{(0)} = T_\xi$ and $T_\eta^{(0)} = T_\eta$ on the boundary, the natural choice for the boundary conditions for the order of theory $n = 1$ would be $T_\eta^{(1)} = T_\xi^{(1)} = 0$ which corresponds, of course, to a plane self-equilibrating system of applied loads. On the other hand, it is a simple matter to show that in general $T_\eta^{(1)}$ and $T_\xi^{(1)}$ as defined by (32) and (33) are not self-equilibrating in the plane sense. Although the portions of (32) and (33) involving $G^{(1)}(\xi,\eta)$ correspond to a self-equilbrating load system in the plane sense by arguments similar to those of Section 3, it can easily be shown that the remaining portions involving the zero'th order solution in general violate the conditions of plane equilibrium. Since the flexibility of the solution rests in the determination of $G^{(1)}(\xi,\eta)$, clearly from a precise standpoint, $G^{(1)}(\xi,\eta)$ is indeterminate.

The difficulty described above is common to most procedures which depend on systematic approximation of a system. The dilemma can be bypassed by introducing an additional approximation. If $G^{(1)}(\xi,\eta)$ is determined on the basis of ensuring prescribed loads over a portion of the boundary only, in many practical problems it can be argued that sufficient accuracy of solution in these regions is assured if the remaining portions of the boundary are handled in only an average sense.

To illustrate the argument above, we consider edge loading of the shell with load-free conditions on the lateral surfaces. It is a straightforward matter to show that $T_\eta^{(1)} + (\xi/R)T_\eta^{(0)}$ and $T_\xi^{(1)} + (\xi/R)T_\xi^{(0)}$ correspond to plane self-equilibrating load systems. On the lateral surfaces of the shell $T_\eta^{(0)}$ and $T_\xi^{(0)}$ vanish, and the conditions on $G^{(1)}(\xi,\eta)$ on the lateral surfaces can be found from (32) and (33). On the ends, $G^{(1)}(\xi,\eta)$ can be adjusted to satisfy the edge loading in an average sense. By appealing to Saint Venant's principle, this approximation should still yield a reliable estimate of the local behavior in the regions of rapidly varying thickness provided these regions are sufficiently removed from the edges.

The axial force resultant $R_\eta^{(1)}$ consistent with (22) for this order theory is defined as

$$R_\eta^{(1)} = 2\pi R \int_{A'}^{B'} \left[T_\eta^{(1)} + (\xi/R)T_\eta^{(0)}\right] ds. \tag{34}$$

This definition of $R_\eta^{(1)}$, in addition to being consistent in form with our plan of successive approximation, enjoys the property of defining the same total resultant axial force independent of the cross section chosen. This last property requires that (34) be independent of path when considered a line integral. It is a straightforward matter to show that (34) meets these requirements.

The corresponding displacements $u^{(1)}$ and $w^{(1)}$ can be carried out in a direct manner; however, the details will not be included here.

V. REMARKS ON ORDERS OF THEORY, n>1

For n>1, the approximations can be analyzed in a manner similar to Sections III and IV. Again the problem for each order can be resolved into the determination of a biharmonic stress function along with a particular solution of (10) in terms of the preceding orders of solution. The determination of the particular solutions becomes increasingly difficult with increasing n. For example, for n = 2 the particular solution requires a solution of the Poisson type differential equation with a righthand side proportional to $F^{(0)}$

The indeterminancy of each of the higher orders of solution resembles the case n = 1. Thus, the plan of enforcing boundary conditions only in the net sense over a portion of the boundary is again necessary in general.

VI. FORMULATION IN TERMS OF COMPLEX VARIABLES

Formulation of the stress functions in terms of analytic functions of a complex variable will now be carried out to enable use of some of the elegant techniques invented for plane problems. Since a departure from the Airy stress function is made, some modification of the conventional complex variable representation will be required. The two orders of solution, n = 0 and n = 1, will now be formulated in terms of the complex variable, $\zeta = \xi + i\eta$.

The biharmonic functions $F^{(0)}(\xi,\eta)$ and $G^{(1)}(\xi,\eta)$ defined in the preceding sections can be represented as

$$F^{(0)}(\xi,\eta) = \operatorname{Im}\left\{\bar{\zeta}\int^{\zeta}\phi^{(0)}(\zeta)d\zeta + \int^{\zeta}\psi^{(0)}(\zeta)d\zeta\right\}, \tag{35}$$

$$G^{(1)}(\xi,\eta) = \operatorname{Im}\left\{\bar{\zeta}\int^{\zeta}\phi^{(1)}(\zeta)d\zeta + \int^{\zeta}\psi^{(1)}(\zeta)d\zeta\right\}, \tag{36}$$

where $\phi^{(n)}(\zeta)$ and $\psi^{(n)}(\zeta)$ are analytic functions of ζ, "Im" denotes the imaginary part of the bracketed expression, and bars denote the complex conjugates. Equations 11 and 35 yield the relations

$$\begin{aligned}
&\sigma_{\xi}^{(0)} + \sigma_{\eta}^{(0)} = 2\left[\phi_{\zeta}^{(0)} + \bar{\phi}_{\bar{\zeta}}^{(0)}\right],\\
&\sigma_{\xi}^{(0)} - \sigma_{\eta}^{(0)} - 2i\,\tau_{\xi\eta}^{(0)} = -2\left[(3-4\nu)\phi_{\zeta}^{(0)} + \bar{\zeta}\,\phi_{\zeta\zeta}^{(0)} + \psi_{\zeta\zeta}^{(0)}\right],\\
&\sigma_{\theta}^{(0)} = \nu\left[\sigma_{\xi}^{(0)} + \sigma_{\eta}^{(0)}\right].
\end{aligned} \tag{37}$$

Similarly, from (30) and (36) along with (35),

$$\begin{aligned}
&\sigma_{\xi}^{(1)} + \sigma_{\eta}^{(1)} = 2\left[\phi_{\zeta}^{(1)} + \bar{\phi}_{\bar{\zeta}}^{(1)}\right] - (1/2R)\left[\zeta\phi_{\zeta}^{(0)} + \bar{\zeta}\bar{\phi}_{\bar{\zeta}}^{(0)} - \psi_{\zeta}^{(0)} - \bar{\psi}_{\bar{\zeta}}^{(0)}\right],\\
&\sigma_{\xi}^{(1)} - \sigma_{\eta}^{(1)} - 2i\tau_{\xi\eta}^{(1)} = -2\left[(3-4\nu)\phi_{\zeta}^{(1)} + \bar{\zeta}\phi_{\zeta\zeta}^{(1)} + \psi_{\zeta\zeta}^{(1)}\right] + \left[\bar{\zeta}\phi_{\zeta}^{(0)} + \psi_{\zeta}^{(0)}\right]/R\\
&\quad + (\zeta + \bar{\zeta})\left[(3-4\nu)\phi_{\zeta}^{(0)} + \bar{\zeta}\phi_{\zeta\zeta}^{(0)} + \psi_{\zeta\zeta}^{(0)}\right]/2R + (1-2\nu)(\phi^{(0)} - \bar{\phi}^{(0)})/R,\\
&\sigma_{\theta}^{(1)} = \nu(\sigma_{\xi}^{(1)} + \sigma_{\eta}^{(1)}) - (1+\nu)\left[\zeta\bar{\phi}_{\bar{\zeta}}^{(0)} + \bar{\zeta}\phi_{\zeta}^{(0)} + \psi_{\zeta}^{(0)} + \bar{\psi}_{\bar{\zeta}}^{(0)}\right]/2R.
\end{aligned} \tag{38}$$

Curvilinear coordinates can now be introduced by introducing appropriate conformal mapping functions of a complex variable. Consider an auxiliary complex plane, the γ-plane, where $\gamma = \epsilon + i\delta$. Let

$$\zeta = \omega(\gamma) \tag{39}$$

denote a conformal mapping of a convenient region in the γ-plane into the region under consideration in the ζ-plane. Then the analyticity of $\phi^{(s)}(\zeta)$ and $\psi^{(s)}(\zeta)$ is preserved when considered as functions of γ due to the analyticity of $\omega(\gamma)$. The argument is well-known in the conventional complex variable formulation and the general details will not be reproduced here.

For reference we shall list several of the more useful relations. If on the arc A'B', we denote

$$d\zeta = ds\, e^{i\alpha} = ds\, e^{i(\pi/2+\beta)}, \tag{40}$$

then,

$$\begin{aligned}
T_\eta^{(0)} + i\, T_\xi^{(0)} &= -e^{i\alpha}\Big[(3-4\nu)\phi_\zeta^{(0)} + e^{-2i\alpha}(\phi_\zeta^{(0)} + \bar{\Phi}_\zeta^{(0)}) + \bar{\zeta}\phi_{\zeta\zeta}^{(0)} + \psi_{\zeta\zeta}^{(0)}\Big], \\
T_\eta^{(1)} + i\, T_\xi^{(1)} &= -e^{i\alpha}\Big[(3-4\nu)\phi_\zeta^{(1)} + e^{-2i\alpha}(\phi_\zeta^{(1)} + \bar{\Phi}_\zeta^{(1)}) + \bar{\zeta}\phi_{\zeta\zeta}^{(1)} + \psi_{\zeta\zeta}^{(1)}\Big]
\end{aligned} \tag{41}$$

$$\begin{aligned}
&- (\zeta + \bar{\zeta})(T_\eta^{(0)} + i\, T_\xi^{(0)})/4R - e^{-i\alpha}(\zeta\bar{\Phi}_\zeta^{(0)} + \bar{\zeta}\phi_\zeta^{(0)} + \psi_\zeta^{(0)} + \bar{\Psi}_\zeta^{(0)})/4R \\
&+ e^{i\alpha}\Big[(1-2\nu)(\phi^{(0)} - \bar{\Phi}^{(0)}) + \bar{\zeta}\phi_\zeta^{(0)} + \psi_\zeta^{(0)}\Big]/2R,
\end{aligned} \tag{42}$$

$$R_\eta^{(0)} = -2\pi R\, R\ell\Big\{4(1-\nu)\phi^{(0)} + \bar{\zeta}\phi_\zeta^{(0)} + \psi_\zeta^{(0)}\Big\}_{A'}^{B'}, \tag{43}$$

$$\begin{aligned}
R_\eta^{(1)} &= -2\pi R\, R\ell\Big\{4(1-\nu)\phi^{(1)} + \bar{\zeta}\phi_\zeta^{(1)} + \psi_\zeta^{(1)}\Big\}_{A'}^{B'} \\
&+ (\pi/2)\, R\ell\Big\{2(3-4\nu)\int^\zeta \phi^{(0)} d\zeta - 4(1-\nu)\zeta\phi^{(0)} - 2(1-2\nu)\bar{\zeta}\phi^{(0)} \\
&+ 2\psi^{(0)} - (\zeta + \bar{\zeta})(\bar{\zeta}\phi_\zeta^{(0)} + \psi_\zeta^{(0)})\Big\}_{A'}^{B'},
\end{aligned} \tag{44}$$

$$2G(u^{(0)} + i\, w^{(0)}) = (3-4\nu)(\phi^{(0)} - \bar{\Phi}^{(0)}) - \zeta\bar{\Phi}_\zeta^{(0)} - \bar{\Psi}_\zeta^{(0)}. \tag{45}$$

VII. SUMMARY

A plan of successive approximations for handling the equations of the three-dimensional elasticity problem for cylindrical shells with regions of rapidly varying thickness is proposed along the conceptual lines of O. Göhner. It is shown that for a restricted but useful class of loadings, the utilisation of plane biharmonic functions can be made; thus, many of the tools of complex variable theory can be introduced.

The argument carried out in the preceding sections points up the requirement for relaxing the precise loading conditions over portions of the shell. This matter is common to all approximate theories of this type, leading to the well-known separation of solution into the so-called interior solutions and boundary layer effects.

Finally, although the generally more difficult three-dimensional problem has been reduced to the determination of plane biharmonic functions, the latter matter can provide considerable difficulties in itself.

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